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COMMENT

Hopping on hierarchical structures and random walking on deterministic fractals

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Abstract. We demonstrate that there exists a definite mapping between hopping on hierarchical structures, with an arbitrary number of sons, and random walking on one new infinitely large family of deterministic loopless fractals. In addition, by explicit claculation of the corresponding spectral dimensions of the introduced fractals, we corroborate the validity of the Einstein diffusion relation for inhomogeneous media.

The classic problem of diffusion in inhomogeneous media has recently been of great interest. On one hand, various models of diffusion on hierarchical structures have been studied (see, for instance, Bachas and Huberman 1987) because of their relevance to numerous physical systems (Havlin and Ben-Avraham 1987). On the other hand, there has been a considerable effort to understand this phenomenon by investigating statistics of random walks on fractal lattices (see, for instance, Vannimenus 1985). Despite many interesting results on both sides, a clear and complete connection between these two approaches has not yet been established.

In a particular case, it has been shown that there is a definite mapping between hopping on the hierarchical structure with given number of sons $m=2$ and random walking on one family of deterministic fractals (Havlin and Weissman 1986). The goal of this comment is, in a way, complementary. Namely, we show that there exists similar mapping between hopping on generalised hierarchical structures with an arbitrary number of sons m (where m is a positive integer) and random walking on one infinitely large family of well defined determinstic loopless fractals. In addition, for this family of fractals, we demonstrate by explicit calculation of the corresponding spectral dimensions d_s that the Einstein diffusion relation is exactly satisfied.

An example ($m=3$) of the generalised hierarchical structures is depicted in figure 1. The structure is constructed iteratively, that is to say the lth iteration comprises m iterations of the $(l-1)$th order, in such a way that the latter are separated by the barriers of height R^{-l}, where $0<R<1$. For arbitrary m it has been shown (Havlin and Weissman 1986) that the fractal dimension of the hopping path is given by

$$d_h = 1+\frac{|\ln R|}{\ln m} \qquad (1)$$

where R determines the height of barriers via the jumping probabilities $w_{k,k+1}=R^l$ $(l>0)$ from the site k to the site $k+1$.

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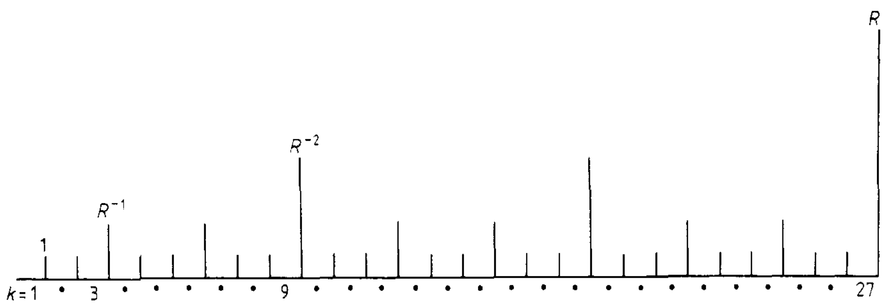

Figure 1. A piece of the generalised hierarchical structure with the number of sons $m = 3$. The hopper (a particle) can hop from a cell denoted by k to one of its nearest neighbours with a probability inversely proportional to the height of the barrier between them.

Here we introduce the new family of fractals. Each member of the family is labelled by an integer b $(2 \leq b \leq \infty)$ and can be obtained from the corresponding generator (see figure 2) by a self-similar growing process (see figure 3). One can notice that the first member of the family is in fact the well known T fractal. In addition, the way the fractals are grown implies the following fractal dimension for arbitrary b

$$d_f(b) = \frac{\ln(2b-1)}{\ln b}. \tag{2}$$

Having introduced the generalised hierarchical structures and the infinite family of the T-like fractals, in what follows we study asymptotic properties of the random walk on the fractals. Specifically, we assume that the random walker is the so-called myopic ant (see, for instance, Havlin and Ben-Avraham 1987). The basic property of the random walk, in general, is its fractal dimension d_w which is determined by d_f and d_s through the Alexander-Orbach relation $d_w = 2d_f/d_s$ (Alexander and Orbach 1982). We will calculate the spectral dimension d_s by performing the exact renormalisation of the random walk master equation (Hilfer and Blumen 1984) for each member of the infinite fractal family. Thus, we first define the appropriate coarse-graining process (see figure 4) and introduce the diagonal $(2b-2) \times (2b-2)$ matrix $\mathbb{D}$, in site labels, whose elements are the coordination numbers of eliminated sites of a fractal generator. Furthermore, we introduce the adjacency $2b \times 2b$ matrix $\mathbb{A}$ defined by $A_{ij} = 1$ if the sites i and j are the nearest neighbours, and $A_{ij} = 0$ if they are not. We denote by $\mathbb{A}_1$

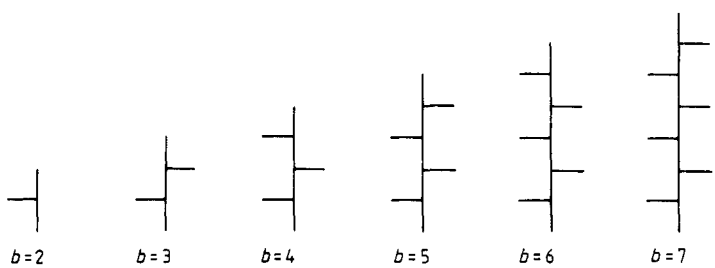

Figure 2. A part of the infinite sequence of generators of the new family of fractals. The original motivation for introducing this family of fractals springs from studying the zinc metal 'trees' (Matsushita *et al* 1985) and the corresponding computer simulations (see, for instance, Meakin 1987).

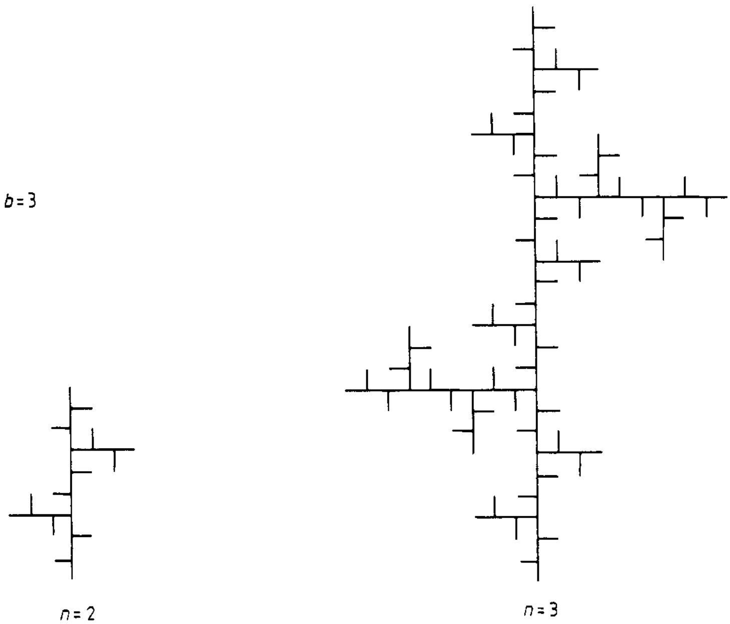

Figure 3. The second ($n = 2$) and the third ($n = 3$) stage of the self-similar growing process of the second member ($b = 3$) of the new fractal family introduced by the sequence of generators depicted in figure 2.

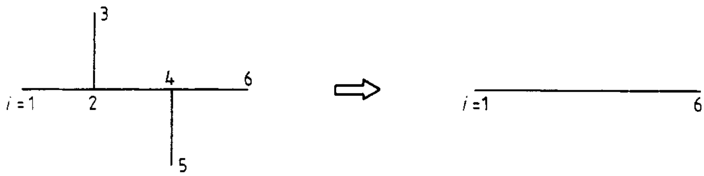

Figure 4. A coarse-graining step in the case of the $b = 3$ member of the fractal family.

the submatrix of $\mathbb{A}$, obtained by deleting all rows and columns in $\mathbb{A}$ which correspond to sites that remain after the coarse-graining step. Finally, we derive matrix $\mathbb{A}_2$ from $\mathbb{A}$ by deleting all rows in $\mathbb{A}$ which correspond to surviving sites and by removing all columns which correspond to eliminated sites in the coarse-graining process.

The introduced matrices furnish the framework for determining exact values of d_s. Following Hilfer and Blumen (1984), we write the spectral dimension of an arbitrary member of the new family of fractals in the form

$$d_s(b) = \frac{2 \ln M}{\ln k} \tag{3}$$

where $M = b^{d_f}$, $k = M/(1-g)$ and $g = [(\mathbb{D} - \mathbb{A}_1)^{-1}\mathbb{A}_2]_{11}$. First, we notice that, for arbitrary b, in the first column of the matrix $\mathbb{A}_2$ only the first element is equal to one and all the others are equal to zero. Thus, in our case, the following relation is valid:

$$g = [(\mathbb{D} - \mathbb{A}_1)^{-1}]_{11}. \tag{4}$$

Next, by studying the sequence of the fractal generators (see figure 2) one can infer the following relation between matrices $(\mathbb{D}-\mathbb{A}_1)_b$ and $(\mathbb{D}-\mathbb{A}_1)_{b-1}$, defined for two successive generators:

$$(\mathbb{D}-\mathbb{A}_1)_b = \begin{pmatrix} 3 & -1 & -1 & 0 & \dots & 0 \\ -1 & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & & & & \\ 0 & 0 & & & & \\ \vdots & \vdots & & & (\mathbb{D}-\mathbb{A}_1)_{b-1} & \\ 0 & 0 & & & & \end{pmatrix} \tag{5}$$

which means that the matrix $(\mathbb{D}-\mathbb{A}_1)_{b-1}$ is a submatrix of $(\mathbb{D}-\mathbb{A}_1)_b$. From this relation it follows that

$$[(\mathbb{D}-\mathbb{A}_1)_b^{-1}]_{11} = \frac{\det(\mathbb{D}-\mathbb{A}_1)_{b-1}}{\det(\mathbb{D}-\mathbb{A}_1)_b} \tag{6}$$

where the abbreviation det has been used to denote determinants of the corresponding matrices. From relation (5) it also follows that

$$\det(\mathbb{D}-\mathbb{A}_1)_b = 2\det(\mathbb{D}-\mathbb{A}_1)_{b-1} - \det(\mathbb{D}-\mathbb{A}_1)_{b-2}. \tag{7}$$

It can be easily checked that $\det(\mathbb{D}-\mathbb{A}_1)_2 = 2$ and that $\det(\mathbb{D}-\mathbb{A}_1)_3 = 3$. Inserting these results into (7) one obtains

$$\det(\mathbb{D}-\mathbb{A}_1)_b = b. \tag{8}$$

The preceding equation, together with relations (2), (3), (4) and (6), provides the explicit and exact formula for the spectral dimension

$$d_s(b) = \frac{2d_f(b)}{d_f(b)+1}. \tag{9}$$

Eventually, using the Alexander-Orbach relation and the derived formula (9), we obtain the random-walk fractal dimension

$$d_w(b) = d_f(b) + 1. \tag{10}$$

One should observe that the above relation between d_w and d_f has been worked out by explict and exact calculations for the entire family of fractals. On the other hand, this relation can be obtained directly, in the case of loopless fractals (Havlin and Ben-Avraham 1987), if the validity of the Einstein diffusion relation for inhomogeneous media is taken for granted. However, although the Einstein relation has been used for inhomogeneous structures, its validity has been frequently questioned (Gefen *et al* 1983, Cates 1984, 1985, Havlin 1985). A general probabilistic interpretation of the meaning of the Einstein relation in the case of inhomogeneous structures, and a specific example, has recently been offered (Hilfer and Blumen 1988). In this context, derivation of formula (10) may be regarded as a corroboration of the validity of the Einstein diffusion relation for an additional family of fractals.

At this point we can establish the connection between random walking on fractals and hopping on hierarchical structures. In the case of our family of fractals, results

(2) and (10) imply that

$$d_w(b) = 1 + \frac{\ln(2b-1)}{\ln b}. \qquad (11)$$

Comparing formulae (1) and (11), one can see that choosing $m = b$ and $R = 1/(2b-1)$ the fractal dimension d_h of the hopping path on a generalised hierarchical structure becomes equal to the fractal dimension d_w of the walking path on the corresponding member of our fractal family. This equivalency is of the same type as that found for a single class of the hierarchical structure ($m = 2$) by Havlin and Weissman (1986).

The established equivalency is pertinent to generalised hierarchical structures with a definite ratio of the barrier heights $1/R$ for a given number of sons m. This equivalence can be easily extended to generalised hierarchical structures with variant R for fixed m, by introducing fractals with bigger maximum coordination numbers z. In the case of all generators presented in figure 2 one can notice that $z = 3$. We can increase z by adding more bonds at the vertices with the maximum coordination number (see figure 5) and let the fractal grow in a self-similar way. Consequently, each fractal of this extended family should be labelled by both b and z, where $3 \leq z \leq 6$ if the fractals are not embedded in the Euclidean spaces with dimension higher than 3. The fractal dimension of such fractals is given by

$$d_f(b, z) = \frac{\ln[b + (b-1)(z-2)]}{\ln b}. \qquad (12)$$

If we now adopt formula (10), the fractal dimension of the random walk on the extended fractals becomes equal to the fractal dimension (1) of the hopping on the

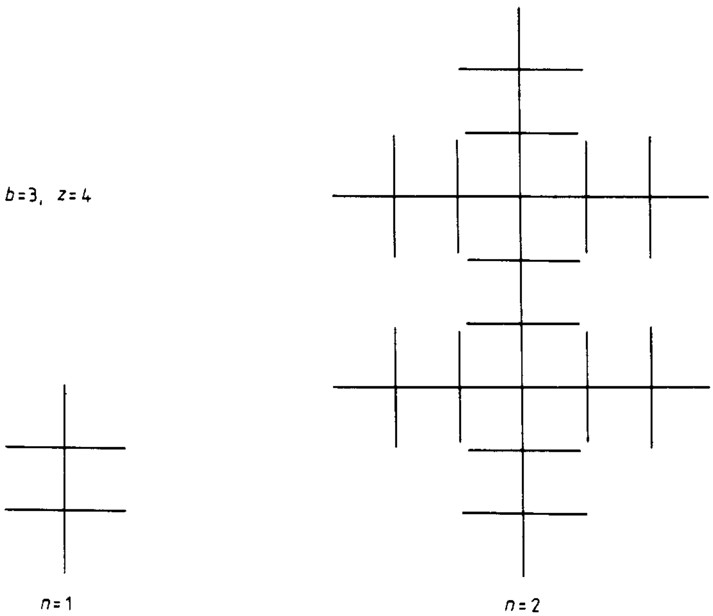

Figure 5. An example of the extension of the fractal family (introduced by the sequence of generators depicted in figure 2) presented in the case $b = 3$ and $z = 4$. The new generator and the second stage ($n = 2$) of growing the corresponding fractal are given.

generalised hierarchical structures with $m = b$ and $R = 1/[b + (b-1)(z-2)]$. Accordingly, different values of b correspond to different numbers of sons, and differenent values of z (for a given b) correspond to variant barrier heights.

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References

Alexander S and Orbach R 1982 *J. Physique Lett.* **43** L625
Bachas C and Huberman B A 1987 *J. Phys. A: Math. Gen.* **20** 4995
Cates M E 1984 *Phys. Rev. Lett.* **53** 926
—— 1985 *Phys. Rev. Lett.* **55** 131
Gefen Y, Aharony A and Alexander S 1983 *Phys. Rev. Lett.* **50** 77
Havlin S 1985 *Phys. Rev. Lett.* **55** 130
Havlin S and Ben-Avraham D 1987 *Adv. Phys.* **36** 695
Havlin S and Weissman H 1986 *J. Phys. A: Math. Gen.* **19** L1021
Hilfer R and Blumen A 1984 *J. Phys. A: Math. Gen.* **17** 537
—— 1988 *Phys. Rev.* A **37** 578
Matsushita M, Hayakawa Y and Sawada Y 1985 *Phys. Rev.* A **32** 3814
Meakin P 1987 *Crit. Rev. Solid State Mat. Sci.* **13** 143
Vannimenus J 1985 *Physics of Finely Divided Matter* ed N Boccara and M Daoud (Berlin: Springer) p 317